MATH 2050C Lecture on 4/22/2020

Reminder: PS 11 & 12 posted, due on Apr 27, May 1 respectively.

(Only count the best 10 out of 12 Problem Sets)

Midterm Statistics: Full Mark: 70 Mean: 54.9 SD: 9.2 Highest: 68

Final: May 5, 2020 8:30 AM – May 6, 2020 8:30 AM

Covers up to (and including) the lectures this Friday.

Uniform Continuity (§5.4)

Recall: Let $f : A \to \mathbb{R}$.

depends on ε

- f cts at $c \in A$ $\iff$ $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x) - f(c)| < \varepsilon$ $\forall\, |x - c| < \delta$.
- f cts on A $\iff$ f is cts at EVERY $c \in A$.

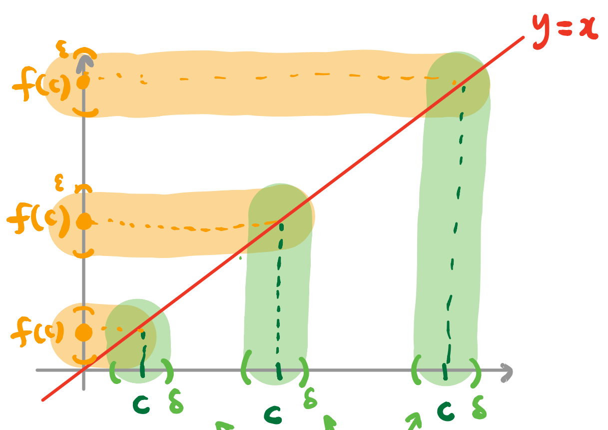
i.e. $\forall\, c \in A$, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x) - f(c)| < \varepsilon$ $\forall\, |x - c| < \delta$

depends on ε AND c

Q: Can one choose $\delta > 0$ independent of c?

A: "Uniform Continuity".

Example 1: $f(x) = x$

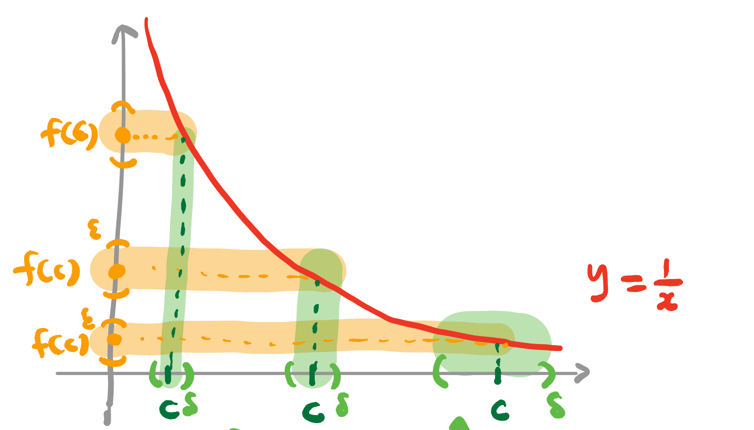


FIX the same $\varepsilon > 0$

the size of these δ-neighborhoods are the same.

unif. cts

Example 2: $f(x) = \dfrac{1}{x}$ $\quad f: (0,\infty) \to \mathbb{R}$



FIX the same $\varepsilon > 0$

these δ-neighborhoods varies when c varies

NOT unif. cts

Def$^{\underline{n}}$: $f: A \to \mathbb{R}$ is uniformly continuous (on A)

iff $\forall \varepsilon > 0$, $\exists \delta \xleftarrow{\text{indep. of } u, v.} = \delta(\varepsilon) > 0$ s.t.

$$|f(u) - f(v)| < \varepsilon \quad \text{whenever } u, v \in A, \ |u - v| < \delta$$

Remarks: (1) unif. cts $\overset{\Rightarrow}{\not\Leftarrow}$ cts on A ($\because$ Take $v = c \in A$)

(2) uniform continuity is a "global" concept, it does NOT make sense to talk about "unif. cts at one pt $c \in A$".

Example 1: $f: \mathbb{R} \to \mathbb{R}$, $f(x) := x$, is uniformly cts. (on $\mathbb{R}$).

Let $\varepsilon > 0$. Choose δ ← dep. on ε only. $:= \varepsilon > 0$. Then, whenever $|u - v| < \delta$, we have

$$|f(u) - f(v)| = |u - v| < \delta = \varepsilon.$$

$\square$

Q: How to show $f: A \to \mathbb{R}$ is NOT unif. cts ?

Nonuniform Continuity Criteria:

$f: A \to \mathbb{R}$ is NOT unif. cts.

$\Leftrightarrow$ $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists u_\delta, v_\delta \in A$ and $|u_\delta - v_\delta| < \delta$

(negation of def$^{\underline{n}}$)

BUT $|f(u_\delta) - f(v_\delta)| \geq \varepsilon_0$.

$\Leftrightarrow$ $\exists \varepsilon_0 > 0$ and seq $(u_n), (v_n)$ in A s.t. $|u_n - v_n| < \frac{1}{n}$ $\forall n \in \mathbb{N}$

(Take $\delta = \frac{1}{n}$)

BUT $|f(u_n) - f(v_n)| \geq \varepsilon_0$. $\forall n \in \mathbb{N}$

Example 2: Show that $f: (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is **NOT** uniformly cts (on $(0,\infty)$).

Pf: Take $\varepsilon_0 := \frac{1}{2} > 0$.

Choose seq. $(u_n) := (\frac{1}{n})$, $(v_n) := (\frac{1}{n+1})$ in $(0,\infty)$.

Note: $|u_n - v_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n}$ $\quad \forall n \in \mathbb{N}$

BUT: $|f(u_n) - f(v_n)| = |n - (n+1)| = 1 > \frac{1}{2} =: \varepsilon_0$. $\quad \forall n \in \mathbb{N}$.

By nonunif cts criteria, f is **NOT** unif. cts on $(0,\infty)$.